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Perturbative construction of self-dual configurations on the torus

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Abstract: We develop a perturbative expansion which allows the construction of non-abelian self-dual SU(2) Yang-Mills field configurations on the four-dimensional torus with topological charge 1/2. The expansion is performed around the constant field strength abelian solutions found by 't Hooft. Next to leading order calculations are compared with numerical results obtained with lattice gauge theory techniques.

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1. Introduction

Self-dual Yang-Mills fields are fascinating mathematical objects that play an important role in both physics and mathematics. In the physics literature they emerged through the introduction of the BPST instanton [1], the minimum action configuration in the sector of topological charge Q equal to one. This triggered the joint effort of physicists and mathematicians in the search for multiinstanton configurations. This had its reward with the ADHM formalism [2], a general set up for the construction of self-dual configurations with vanishing field strength at infinity (which, from the topological point of view, corresponds to fields living on a four-sphere). However, there are certain instances in which other types of boundary conditions are relevant. For example, in considering finite temperature field theory, one is interested in studying configurations which are periodic in thermal time. This periodicity might also be used as a device to study monopole-like objects. In the same fashion, additional periodicities might have other uses and interpretations, as argued in ref. [3]. This justifies an initial interest in the study of self-dual configurations which are periodic in all euclidean space-time directions. Geometrically, it corresponds to the study of self-dual gauge fields on the torus. It might be considered as the next step after the case of gauge fields on the sphere. The study of gauge fields on the torus (see [4] for a review of this topic) brings in a new topological richness, whose appearance, physical interpretation and usefulness was put forward by 't Hooft [5]. In addition, periodic self-dual configurations constitute a simple mathematical example of

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dense multiinstanton configurations, which, as advocated in ref. [6], might turn out to be a better description of the confining vacuum than other dilute multiinstanton pictures.

The construction of self-dual configurations on the torus has met very limited success. For particular values of the torus sizes, there is a class of solutions which is known since early times [7, 8]. These configurations have constant field strength, and they are, in some sense, of an abelian nature: all the spatial components of the electric and magnetic fields are parallel in colour space. For other torus sizes, we know, through numerical techniques [9, 10, 3], that the solutions are quite different. They turn out to be lumpy and very non-abelian: different electric field components are mutually orthogonal in colour space at the center of the lump. Fortunately, there has recently been substantial progress in studying the mixed situation of instanton configurations which are periodic in only a few directions. In particular, in the case of only one compactified direction, corresponding to the physical situation of finite temperature, the most general solution of topological charge one, the caloron, has been found [11]. Considerable progress has also been achieved for the case of gauge fields on $T^2 \times \mathbb{R}^2$ (doubly periodic instantons), by the work of mathematicians [12] and physicists [3, 13, 14]. In these developments a crucial role is played by the Nahm transform, a duality transformation which maps self-dual configurations on tori with dual sizes [15, 16]. For the $T^3 \times \mathbb{R}$ case, the calculation of the abelian Nahm dual of the charge one instanton [17] represents a step forward. The caloron, the doubly periodic instantons and the $T^3 \times \mathbb{R}$ instanton can be viewed as limiting cases of configurations on the torus where either three, two or one directions are taken to be very large with respect to the others.

This paper is a step towards an analytical understanding of non-constant field strength self-dual configurations on the torus. The strategy is to consider torus sizes which depart only slightly from those in which there exists a self-dual constant field strength solution. Our construction is based upon perturbing around the constant solution (at a torus size in which it is not self-dual), and imposing self-duality to the resulting configuration. A systematic perturbative expansion arises that allows the construction of the self-dual solution. We show that the solution exists order by order and is unique up to gauge transformations and space-time translations. This is done in section 2. Our approach is intimately related to the study of van Baal [8], who considered perturbations around constant field strength solutions.

In section 3 we proceed to compare the lowest non-trivial order results obtained from our expansion with the exact solution obtained by numerical methods. This serves to quantify the rate of convergence of the series for various torus sizes. Finally, in section 4, we investigate the interplay of our perturbative expansion with the Nahm transform. The paper is closed by section 5, where the conclusions and possible extensions are presented.

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2. The construction

Let us consider SU(2) gauge fields living on a torus of size $l_0 \times l_1 \times l_2 \times l_3$. Under a translation by one of the periods, the gauge potentials and fields transform by a gauge transformation:

$$A_\nu(x + e_\mu) = [\Omega_\mu(x)] A_\nu(x)\,, \tag{2.1}$$

where e_μ is a 4-vector of length l_μ along the μ-th direction, and Ω_μ are the twist matrices. The compatibility conditions of the previous equations are:

$$\Omega_\mu(x + e_\nu)\, \Omega_\nu(x) = \exp\{\pi \imath n_{\mu\nu}\}\, \Omega_\nu(x + e_\mu)\, \Omega_\mu(x)\,, \tag{2.2}$$

where the elements of the antisymmetric twist tensor $n_{\mu\nu}$ are integers defined modulo 2. In what follows we will choose $n_{03} = n_{12} = -n_{30} = -n_{21} = 1$, with the remaining components being zero. For the twist matrices we will take

$$\Omega_\mu(x) = \exp\left\{\imath \frac{\pi}{2} n_{\mu\nu} \frac{x_\nu}{l_\nu} \tau_3\right\}, \tag{2.3}$$

which is consistent with our choice of twist tensor. The symbols τ_i label the Pauli matrices.

For torus sizes such that $l_0 l_3 = l_1 l_2$, there exist self-dual configurations satisfying the previous boundary conditions and having constant field strength. What we will do is to consider a slight deviation from this situation controlled by the parameter:

$$\Delta = \frac{l_0 l_3 - l_1 l_2}{\sqrt{V}}\,, \tag{2.4}$$

where $V = l_0 l_1 l_2 l_3$ is the torus volume. Without loss of generality we can assume that Δ is positive. In this case there exists a constant field strength configuration with vector potential:

$$B_\mu(x) = -\frac{\pi}{2} n_{\mu\nu} \frac{x_\nu}{l_\mu l_\nu} \tau_3\,. \tag{2.5}$$

This gauge potential gives rise to a field strength of the form $G_{\mu\nu}\tau_3$, where:

$$G_{\mu\nu} = \pi \frac{n_{\mu\nu}}{l_\mu l_\nu}\,, \tag{2.6}$$

The only non-zero components are G_{03} and G_{12}, which become of equal magnitude at $\Delta = 0$, rendering the solution self-dual. With our choice of twist the constant field strength configuration has topological charge $Q = 1/2$ and, for $\Delta = 0$, total action $4\pi^2$.

Now, let us consider perturbing around this gauge potential:

$$A_\mu(x) = B_\mu(x) + S_\mu(x)\, \tau_3 + W_\mu(x)\, \tau_+ + W^*_\mu(x)\, \tau_-\,, \tag{2.7}$$

where we have decomposed the additional field into different colour components. The matrices $\tau_\pm = \frac{1}{2}(\tau_1 \pm \imath\tau_2)$ are standard. The boundary conditions on the gauge fields translate into the real function $S_\mu(x)$ being periodic on the box, and the complex function $W_\mu(x)$ satisfying

$$W_\rho(x+e_\mu) = \exp\left\{\imath\pi\, n_{\mu\nu}\frac{x_\nu}{l_\nu}\right\} W_\rho(x)\,. \tag{2.8}$$

We will make the following gauge choice, consistent with these boundary conditions (the background field gauge):

$$\partial_\mu\, A_\mu(x) - \imath[B_\mu(x), A_\mu(x)] = 0\,. \tag{2.9}$$

We will now demand that the resulting gauge field is self-dual

$$\begin{aligned} \mathbf{F}_{\mu\nu}(x) - \widetilde{\mathbf{F}}_{\mu\nu}(x) &= 0\,, \\ \widetilde{\mathbf{F}}_{\mu\nu}(x) &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{F}_{\rho\sigma}(x)\,, \qquad \text{with } \epsilon_{0123} = 1\,, \end{aligned} \tag{2.10}$$

which will be interpreted as equations for the functions $S_\mu(x)$ and $W_\mu(x)$. The best way to express these equations, together with the gauge fixing condition, is to use the matrices $\sigma_\mu \equiv (\mathbf{I}, -\imath\vec{\tau})$ and $\overline{\sigma}_\mu \equiv (\mathbf{I}, \imath\vec{\tau}) = \sigma_\mu^\dagger$. These matrices satisfy:

$$\overline{\sigma}_\mu\sigma_\nu = \overline{\eta}^\alpha_{\mu\nu}\sigma_\alpha\,, \tag{2.11}$$

where $\overline{\eta}^\alpha_{\mu\nu}$ is the 't Hooft symbol, such that $\overline{\eta}^0_{\mu\nu} = \delta_{\mu\nu}$ and the $\overline{\eta}^i_{\mu\nu}$ are a basis of the antiself-dual tensors. Now contracting $\mathbf{F}_{\mu\nu}(x)$ with $\overline{\sigma}_\mu\sigma_\nu$, we project out the self-dual part. Hence, we might rewrite equation (2.10) as follows:

$$\overline{\partial} S = -\frac{\lambda}{2}\widehat{G} + \frac{\imath}{2}(W_c^\dagger W_c - W^\dagger W) \tag{2.12}$$

$$\overline{D} W = \imath(S^\dagger W - W_c^\dagger S)\,, \tag{2.13}$$

where $S = S_\mu(x)\sigma_\mu$ and $W = W_\mu(x)\sigma_\mu$ are 2×2 matrices, $S^\dagger$, $W^\dagger$ their adjoints and the parameter λ is equal to 1. The matrix W_c is the charge conjugate:

$$W_c = \tau_2 W^* \tau_2\,. \tag{2.14}$$

The matrix $\widehat{G} \equiv G_{\mu\nu}\overline{\sigma}_\mu\sigma_\nu$ is given by:

$$\widehat{G} = 2\pi\imath\frac{\Delta}{\sqrt{V}}\tau_3\,. \tag{2.15}$$

It vanishes when $\Delta = 0$, since then the constant field strength configuration is self-dual. Finally, we define the following differential operators:

$$\partial = \sigma_\mu\partial_\mu\,, \tag{2.16}$$

$$\overline{\partial} = \overline{\sigma}_\mu\partial_\mu\,, \tag{2.17}$$

$$D = \sigma_\mu D_\mu\,, \tag{2.18}$$

$$\overline{D} = \overline{\sigma}_\mu D_\mu\,, \tag{2.19}$$

$$D_\mu = \partial_\mu + \imath\pi\frac{x_\nu n_{\mu\nu}}{l_\mu l_\nu}\,, \tag{2.20}$$

Since at $\Delta = 0$ the correction terms W and S vanish, we can think of Δ as a perturbation parameter. Rigorously speaking this is not quite so, because Δ depends on the torus sizes, and they enter also in the boundary conditions. To keep a truly perturbative parameter in our expansion we introduced in eq. (2.12) the parameter λ, whose interpretation will be clear later. Our goal will be to solve eqs. (2.12)–(2.13) for arbitrary values of λ, as a perturbative expansion in powers of $\lambda\Delta$. For that we have to expand the unknown matrices S,W in powers of $\sqrt{\lambda\Delta}$. We see that the equations are consistent with W carrying odd powers and S even powers:

$$W = \frac{1}{\sqrt{\lambda\,\Delta}} \sum_{k=1}^{\infty} (\lambda\,\Delta)^k W^{(k)} \tag{2.21}$$

$$S = \sum_{k=1}^{\infty} (\lambda\,\Delta)^k S^{(k)} \,. \tag{2.22}$$

In the following paragraphs we will show that it is possible to solve the set of equations (2.12)–(2.13) order by order in $\sqrt{\lambda\Delta}$. Finally, setting $\lambda = 1$ one recovers the solution of the self-duality condition. However, even for this value higher orders in the expansion are suppressed by powers of $\sqrt{\Delta}$. Thus, as we will verify later, we expect the first few terms of the expansion to approximate the self-dual solutions for small values of Δ.

On the other hand, the solution for arbitrary λ can be interpreted as the solution of the following modified self-duality equation:

$$\mathbf{F}_{\mu\nu}(x) - \widetilde{\mathbf{F}}_{\mu\nu}(x) = (1-\lambda)(G_{\mu\nu} - \widetilde{G}_{\mu\nu})\tau_3 \,. \tag{2.23}$$

For $\lambda = 0$ the constant field strength configuration is a solution, while for $\lambda = 1$ we recover the self-duality equations.

Now let us address solving the equations order by order in λ. Notice first that $W^{(1)}$ satisfies the equation:

$$\overline{D} W^{(1)} = 0 \,. \tag{2.24}$$

This equation has non-zero regular solutions, as can be deduced from the index theorem. As we will see later the general solution has the form:

$$W^{(1)} = \Psi(x) \begin{pmatrix} K^{(1)} & Q^{(1)} \\ 0 & 0 \end{pmatrix} , \tag{2.25}$$

where $K^{(1)}$ and $Q^{(1)}$ are two arbitrary complex numbers and $\Psi(x)$ is a function whose explicit form will be given later. Having seen that eq. (2.13) has a solution for $k = 1$, let us address the question of whether it also has a solution for all values of k. For that purpose one has to investigate the adjoint of $\overline{D}$, and see whether its kernel is null or not. Indeed, it is easy to see that the kernel of the adjoint vanishes, and hence eq. (2.13) has a regular solution no matter what the left-hand side is, provided it is regular. Explicitly, one such solution is given by:

$$W^{(k)} = D\,U^{(k)} \tag{2.26}$$

where

$$\overline{D}D\,U^{(k)} = \imath \sum_{l=1}^{k-1} \left(S^{(k-l)\dagger} W^{(l)} - W_c^{(l)\dagger} S^{(k-l)}\right). \tag{2.27}$$

The existence of a unique solution $U^{(k)}$ to eq. (2.27) follows from the invertibility of the operator $\overline{D}D$. In terms of this particular solution, and using (2.24)–(2.25), the most general solution of eq. (2.13) is given by:

$$W^{(k)} = D\,U^{(k)} + \Psi(x) \begin{pmatrix} K^{(k)} & Q^{(k)} \\ 0 & 0 \end{pmatrix}, \tag{2.28}$$

where $K^{(k)}$ and $Q^{(k)}$ are complex constants.

Now let us study the solution of equation (2.12) order by order in λ. It is easy to see that both the left and the right hand sides are periodic functions in the box. They can hence be expanded in Fourier series. The solution can be obtained by equating the corresponding Fourier coefficients. However, notice that the left-hand side has no constant term. Hence, if the right-hand has a non-zero constant Fourier term, the equation has no solution. This can be expressed more formally by saying that the kernel of ∂ is nontrivial. What we will now show is that in solving the equation for $S^{(k)}$ it is possible to fix the constants $K^{(k)}$ and $Q^{(k)}$ appearing in the solution to (2.13) to order k, by the condition that the lowest Fourier component of the right-hand side vanishes. For that we have to explore the effect of the replacement (2.28) in eq. (2.12) to order $(\lambda\,\Delta)^k$. The dependence of the right-hand side of eq. (2.12) on $K^{(k)}$ and $Q^{(k)}$ to this order, is contained in the following term:

$$|\Psi(x)|^2 (c_3 \tau_3 + c_+ \tau_+ + c_+^* \tau_-)\,, \tag{2.29}$$

where $c_3 = 2\,\mathrm{Re}(Q^{(k)} Q^{(1)\,*} - K^{(k)} K^{(1)\,*})$ and $c_+ = 2(K^{(k)\,*} Q^{(1)} + K^{(1)\,*}\, Q^{(k)})$. The symbol Re denotes the real part of its complex argument. By choosing $K^{(k)}$, $Q^{(k)}$ appropriately the term within parenthesis can be made equal to an arbitrary hermitean, traceless 2×2 matrix. Since the lowest Fourier coefficient (the constant one) of $|\Psi(x)|^2$ is non-zero, the constants can be chosen such that the whole right hand side of eq. (2.12) has vanishing constant Fourier term. Actually, this fixes 3 of the 4 real parameters which enter $K^{(k)}$, $Q^{(k)}$. The remaining one corresponds to the symmetry associated to global colour rotations in the 1-2 plane, which leave B_μ invariant (we will comment upon this property on the next paragraph).

Having shown that the solution of our set of equations exists order by order in our expansion in $\sqrt{\lambda\Delta}$, we have now to analyse uniqueness. Indeed, on general grounds we know that the solution is non-unique. This fact is associated to the existence of transformations which change one solution into other. We already mentioned one: global gauge transformations of a certain kind. These are the residual gauge transformations that are not gauge fixed by eq. (2.9). They are associated with the freedom to multiply any solution matrix W by a constant phase. The other transformations are space-time translations (followed by an appropriate gauge transformation to preserve the gauge fixing condition). This latter symmetry manifests itself under the form of a non-uniqueness for the solutions of eq. (2.12) to any order k: notice that we are free to add an arbitrary constant matrix to S in the left-hand side of the equation, which would entail the four real parameters associated to a translation.

The best strategy in solving the equations is to fix a unique solution by constraining these transformations. This we will do by imposing the following additional conditions:

$$\mathrm{Re}(W_{12}(x=0)) = |W_{12}(x=0)| \tag{2.30}$$

$$\int dx\, S(x) = 0\,, \tag{2.31}$$

(If $W_{12}(x=0)=0$ we take $\mathrm{Re}(W_{11}(x=0)) = |W_{11}(x=0)|$). It is now completely clear that the procedure leads to a unique solution order by order in $\sqrt{\lambda}$.

It is useful to derive expressions for the field strength tensor itself. Just as we did for the vector potential we might expand in colour components:

$$\mathbf{F}_{\mu\nu}(x) \;= F^{(3)}_{\mu\nu}(x)\,\tau_3 + F^{(+)}_{\mu\nu}(x)\,\tau_+ + F^{*\,(+)}_{\mu\nu}(x)\,\tau_-\,. \tag{2.32}$$

Now since the field is self-dual, we might contract it with the matrices $\sigma_\mu\overline{\sigma}_\nu$ to obtain a traceless hermitean matrix combining the three spatial directions:

$$\begin{aligned}
\mathcal{F}^{(+)} &\equiv \frac{-\imath}{4} F^{(+)}_{\mu\nu}(x)\,\sigma_\mu\overline{\sigma}_\nu \\
&= \frac{-\imath}{2} D W_c^\dagger - \frac{1}{2}(S\,W_c^\dagger - W\,S^\dagger) \qquad (2.33)\\
\mathcal{F}^{(3)} &\equiv \frac{-\imath}{4} F^{(3)}_{\mu\nu}(x)\,\sigma_\mu\overline{\sigma}_\nu \\
&= \frac{\pi}{2}\left(\frac{l_0 l_3 + l_1 l_2}{V}\right)\tau_3 - \frac{\imath}{2}\partial S^\dagger - \frac{1}{4}(W\,W^\dagger - W_c\,W_c^\dagger)\,. \qquad (2.34)
\end{aligned}$$

Having set up the full procedure for calculating the potentials and fields in powers of λ, let us now exemplify it by computing the first terms in this expansion. These results will be used in the next section. The starting point is the equation for $W^{(1)}$ (eq. (2.24)). We mentioned previously what is the form of the solution. Let us for the moment skip the proof and also the determination of $\Psi(x)$ and proceed. The next step is to look at the equation for $S^{(1)}$. As mentioned in the general case, both the left and right-hand sides can be expanded in Fourier coefficients. The condition that the constant coefficient of the right hand side vanishes imposes a constraint on $K^{(1)}$ and $Q^{(1)}$:

$$\begin{aligned} Q^{(1)\,*}K^{(1)} &= 0 \\ |Q^{(1)}|^2 - |K^{(1)}|^2 &= \frac{2\pi}{\sqrt{V}}\,, \end{aligned} \tag{2.35}$$

where we have fixed the normalisation of $\Psi(x)$, such that its constant Fourier coefficient is equal to one. The previous equations lead to

$$K^{(1)} = 0\,; \qquad Q^{(1)} = \frac{\sqrt{2\pi}}{V^{1/4}}\,, \tag{2.36}$$

where we have used (2.30).

Now the equation for $S^{(1)}$ reads:

$$\bar{\partial} S^{(1)} = \frac{\imath\pi}{\sqrt{V}}(|\Psi(x)|^2 - 1)\tau_3\,. \tag{2.37}$$

This can be solved together with eq. (2.31) to give:

$$S^{(1)} = \frac{\imath\pi}{\sqrt{V}}(\partial h)\tau_3\,, \tag{2.38}$$

where $h(x)$ is a periodic function on the box, solution of the equation:

$$\Box h(x) = |\Psi(x)|^2 - 1 \tag{2.39}$$

and $\Box$ is the 4-dimensional laplacian. The previous equation can be solved by expanding both sides in Fourier series and equating.

Let us now work out the details of the solution to eq. (2.24). For future purposes we will consider a more general equation:

$$\overline{D_q}\,\varphi(x) = 0\,, \tag{2.40}$$

where φ is a two component vector. The operator $\overline{D_q}$ is given by:

$$\overline{D_q} = \overline{\sigma}_\mu \left(\partial_\mu + \imath\pi\, q \frac{x_\nu n_{\mu\nu}}{l_\mu l_\nu}\right)\,, \tag{2.41}$$

where q is a constant. The vector of functions $\varphi(x)$ is required to satisfy the boundary condition:

$$\varphi(x + e_\mu) = \exp\left\{\imath\pi\, q\, n_{\mu\nu}\frac{x_\nu}{l_\nu}\right\}\varphi(x)\,, \tag{2.42}$$

which is only consistent for integer q. It is easy to see that the operator $\overline{D_q}$ preserves this boundary condition.

Now it is seen that locally a solution of eq. (2.40) takes the form:

$$\begin{pmatrix} \widetilde{\varphi}_q(x)\,\kappa_+(u_0,u_1) \\ (\widetilde{\varphi}_q(x))^{-1}\,\kappa_-(u_0^*,u_1^*) \end{pmatrix}, \tag{2.43}$$

where we have introduced complex coordinates:

$$u_\mu = \frac{1}{l_\mu}(x_\mu + \imath\, n_{\mu\nu} x_\nu) \tag{2.44}$$

and u_μ^* are the complex conjugates. These coordinates are not independent, and satisfy:

$$u_\mu = \frac{\imath}{l_\mu} n_{\mu\nu} u_\nu l_\nu\,. \tag{2.45}$$

We might for future benefit introduce the complex constants:

$$\tau_\mu = \frac{\imath}{l_\mu} |n_{\mu\nu} l_\nu|\,. \tag{2.46}$$

The function $\widetilde{\varphi}_q$ is given by:

$$\widetilde{\varphi}_q(x) = \exp\left\{-\frac{\pi q}{2 l_0 l_3}(x_0^2 + x_3^2) - \frac{\pi q}{2 l_1 l_2}(x_1^2 + x_2^2)\right\}. \tag{2.47}$$

The boundary conditions eq. (2.42) impose constraints on the value on the holomorphic and anti-holomorphic functions $\kappa_\pm$:

$$\begin{aligned} \kappa_+(x+e_\mu) &= \exp\left\{\pi q \imath \frac{(u_\mu + 1/2)}{\tau_\mu}\right\} \kappa_+(x) \\ \kappa_-(x+e_\mu) &= \exp\left\{-\pi q \imath \frac{(u_\mu^* + 1/2)}{\tau_\mu}\right\} \kappa_-(x)\,. \end{aligned} \tag{2.48}$$

Choosing u_0 and u_1 as our two independent complex variables, we might write:

$$\begin{aligned} \kappa_+(u_0,u_1) &= \exp\left\{\pi q \imath \left(\frac{u_0^2}{2\tau_0} + \frac{u_1^2}{2\tau_1}\right)\right\} \widetilde{\kappa}_+(u_0,u_1) \\ \kappa_-(u_0^*,u_1^*) &= \exp\left\{-\pi q \imath \left(\frac{u_0^{*2}}{2\tau_0} + \frac{u_1^{*2}}{2\tau_1}\right)\right\} \widetilde{\kappa}_-(u_0^*,u_1^*)\,. \end{aligned} \tag{2.49}$$

The functions $\widetilde{\kappa}_\pm$ are periodic in their arguments with period 1 and satisfy:

$$\begin{aligned} \widetilde{\kappa}_+(u_0+\tau_0,u_1) &= \exp\{-2\pi q \imath u_0 - \pi q \imath \tau_0\} \widetilde{\kappa}_+(u_0,u_1) \\ \widetilde{\kappa}_+(u_0,u_1+\tau_1) &= \exp\{-2\pi q \imath u_1 - \pi q \imath \tau_1\} \widetilde{\kappa}_+(u_0,u_1)\,. \end{aligned} \tag{2.50}$$

For $q=1$ these are precisely the conditions satisfied by the Riemann θ function [18]. Actually, up to a multiplicative constant, this function is the only (regular) holomorphic function satisfying these boundary conditions. Similarly, one obtains that the

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equation for $\widetilde{\kappa}_-$ has no regular solutions. Hence, for the $q = 1$ case we have arrived at the solution given in eq. (2.25), and determined the expression for the function $\Psi(x)$:

$$\Psi(x) = \sqrt{\frac{4l_3l_2}{l_0l_1}} \exp\left\{-\frac{\pi}{l_0l_3}(x_3^2 - \imath x_3x_0) - \frac{\pi}{l_1l_2}(x_2^2 - \imath x_2x_1)\right\} \theta(u_0, \tau_0)\, \theta(u_1, \tau_1)\,. \tag{2.51}$$

The multiplicative factor preceding the right hand side of the previous expression is determined by the condition that the lowest Fourier coefficient of $|\Psi(x)|^2$ is unity.

3. Comparison with numerical results

In the previous section we have set up a general procedure to construct the form of the gauge potentials and field strengths for SU(2) self-dual solutions on the torus with twist tensor $n_{03} = n_{12} = 1$. The result is an expansion in powers of $\sqrt{\Delta}$, where Δ is defined in eq. (2.4). However, we do not have an analytical estimate of the size of the coefficient. Our purpose in this section is to test the rate of convergence of the expansion by comparing the results obtained from the first non-trivial order with the exact result as obtained by numerical methods on the lattice.

We will restrict to the analysis of the gauge invariant traces $\mathrm{Tr}(\mathbf{E}_i\mathbf{E}_j)$, where $\mathbf{E}_i = \mathbf{F}_{0i}$ are the electric fields. To lowest order in our expansion, we fall into the constant field strength configuration, and the only non-zero gauge invariant trace is $\mathrm{Tr}(\mathbf{E}_3^2)$. The next correction is order Δ and vanishes for $\mathrm{Tr}(\mathbf{E}_1\mathbf{E}_2)$. It also predicts that $\mathrm{Tr}(\mathbf{E}_1^2) = \mathrm{Tr}(\mathbf{E}_2^2)$. Making use of the general formulas (2.33), (2.34) and substituting the explicit form of $S^{(1)}$ and $W^{(1)}$ (eqs. (2.38), (2.25)) one arrives at the following result, valid to order Δ:

$$\mathrm{Tr}(\mathbf{E}_1^2(x)) = \mathrm{Tr}(\mathbf{E}_2^2(x)) = \Delta\frac{\pi}{\sqrt{V}}\left|D_0\Psi(x)\right|^2, \tag{3.1}$$

$$\begin{aligned}\mathrm{Tr}(\mathbf{E}_3^2(x)) = {} & \frac{\pi^2}{2}\left(\frac{l_0l_3 + l_1l_2}{V}\right)^2 \times \\ & \times\left\{1 - \Delta\left(\frac{2\sqrt{V}}{l_0l_3 + l_1l_2}\right)\left(1 + 2(\partial_0^2 + \partial_3^2)h(x)\right)\right\},\end{aligned} \tag{3.2}$$

$$\mathrm{Tr}(\mathbf{E}_1(x)\mathbf{E}_3(x)) = -\Delta\pi^2\left(\frac{l_0l_3 + l_1l_2}{V^{3/2}}\right)(\partial_0\partial_2 + \partial_1\partial_3)\, h(x)\,, \tag{3.3}$$

$$\mathrm{Tr}(\mathbf{E}_2(x)\mathbf{E}_3(x)) = \Delta\pi^2\left(\frac{l_0l_3 + l_1l_2}{V^{3/2}}\right)(\partial_0\partial_1 - \partial_2\partial_3)\, h(x)\,, \tag{3.4}$$

where Ψ and h have been defined in section 2 (eqs. (2.51) and (2.39)). Using the standard representation of Riemann's theta function [18]:

$$\theta(u, \tau) = \sum_{n\in\mathbf{Z}} \exp\{2\pi\imath nu + \pi\imath n^2\tau\} \tag{3.5}$$

one can easily obtain the Fourier coefficients of all the functions appearing in eqs. (3.1)–(3.4). As for the numerical comparison, summing the first few hundred terms of the Fourier expansion allows to compute these functions with negligible errors. It is also extremely simple to use these Fourier coefficients to integrate analytically over some of the four real coordinates, to arrive at a quantity better suited for graphically displaying the comparison.

A numerical approximation to the exact solutions of the self-duality equations, with which the results coming from the perturbative approximation that we have developed are to be compared, can be constructed by means of standard lattice gauge theory techniques [9, 10]. We will use for this purpose an $\varepsilon = 0$ overimproved cooling procedure, that was found in previous works [10] to be able to produce very accurate approximants to continuum self-dual fields. In particular, it allows to extract the exact values of the gauge invariant densities $\mathrm{Tr}(\mathbf{E}_i\mathbf{E}_j)$ under concern up to $\mathcal{O}(a^4)$ corrections, a being the lattice spacing (whose precise definition we will discuss below).

To explore the accuracy of the next to leading term in the perturbative expansion with varying values of the remaining parameters, we will consider tori of lengths ($l_0 = l_t(1+\epsilon), l_1 = l_t, l_2 = l_s, l_3 = l_s$). The results will then depend on the perturbative parameter $\Delta = \epsilon/\sqrt{1+\epsilon}$, and on the ratio l_s/l_t measuring the degree of spatial asymmetry of the torus. Thus, by keeping $0 \leq \epsilon \ll 1$ we will remain within the perturbative regime. On the other hand, we will vary the ratio l_s/l_t between 1 (the more symmetrical case) and 0, where the θ functions and their derivatives entering into the analytical expressions are well described by polynomials times gaussians.

In each case, a numerical solution is obtained in a lattice with a number of points $L_\mu = l_\mu/a$ along direction μ. To define the lattice spacing a, we need to fix a unit. In our case we take $l_1 l_2 = 1$. This is justified by noticing from the expression for Ψ in eq. (2.51) that the region having nontrivial structure in the action density at nonzero Δ is of size $\sqrt{l_1 l_2}$.

We will present the results of the comparison of the analytical results eqs. (3.1)–(3.4) with their numerical counterparts, for three different configurations, having different values of Δ and l_s/l_t. The lattice sizes that we will use, together with their associated Δ and l_s/l_t values, are detailed in table 1.

Lattice	Size	Δ	l_s/l_t
A	$13 \times 12 \times 12 \times 12$	0.080064	1.00
B	$21 \times 20 \times 8 \times 8$	0.048795	0.40
C	$41 \times 40 \times 6 \times 6$	0.024693	0.15

Table 1: Lattices used in the comparison, and their associated Δ and l_s/l_t values.

In figures 1, 2 and 3 we show the numerical and perturbative results for the integrated electric field densities $\Phi^{(2)}_{33}(x_0,x_1) \equiv \int \mathrm{d}x_2\mathrm{d}x_3\,\mathrm{Tr}(\mathbf{E}_3^2(x))$ and $\Phi^{(2)}_{11}(x_0,x_1) = \Phi^{(2)}_{22}(x_0,x_1) \equiv \int \mathrm{d}x_2\mathrm{d}x_3\,\mathrm{Tr}(\mathbf{E}_1^2(x))$, and $\Phi^{(2)}_{23}(x_0,x_1) \equiv \int \mathrm{d}x_2\mathrm{d}x_3\,\mathrm{Tr}(\mathbf{E}_2(x)\mathbf{E}_3(x))$, respectively (notice that $\Phi^{(2)}_{13}$ vanishes at the present perturbative order, despite the fact that $\mathrm{Tr}(\mathbf{E}_1(x)\mathbf{E}_3(x))$ does not, because of the particular form of the expression for this

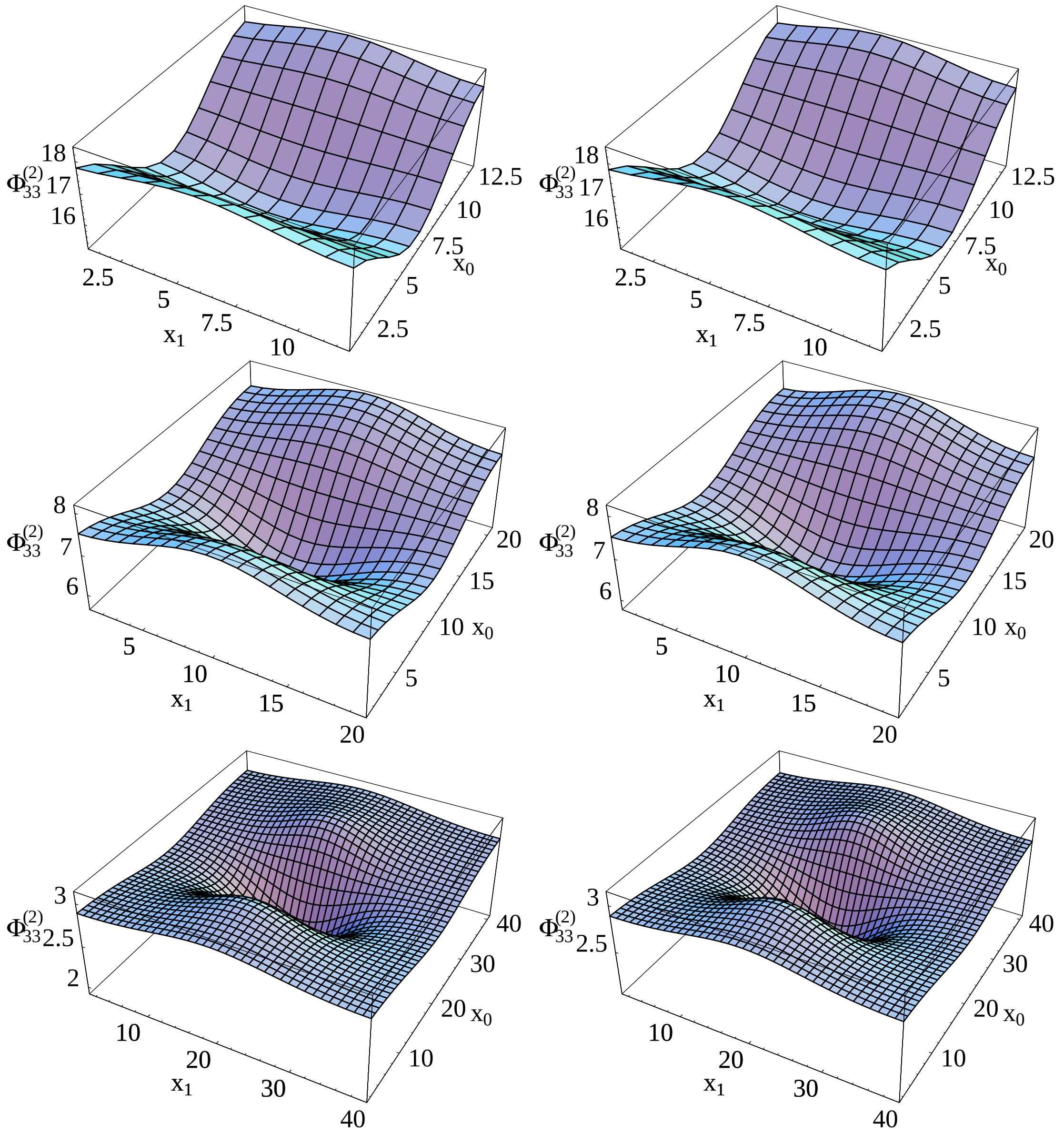

Figure 1: Invariant densities $\Phi_{33}^{(2)}$ for the configurations A, B, C of table 1, from top to bottom, are shown. Plots in the left column display the analytical perturbative result, and plots in the right column display the exact (numerical) result.

latter quantity). The qualitative agreement is clearly good. The main features of the exact solution are present in the analytical expression. It is possible to obtain a graphical quantitative measure of the comparison by integrating the previous densities over an additional coordinate to yield the time profiles $\Phi_{33}^{(1)}(x_0) \equiv \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \mathrm{Tr}(\mathbf{E}_3^2(x))$ and $\Phi_{11}^{(1)}(x_0) = \Phi_{22}^{(1)}(x_0) \equiv \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \mathrm{Tr}(\mathbf{E}_1^2(x))$ (similarly to what happened with $\Phi_{13}^{(2)}$, $\Phi_{23}^{(1)}$ vanishes despite $\Phi_{23}^{(2)}$ does not). The comparison for these quantities is displayed in figure 4.

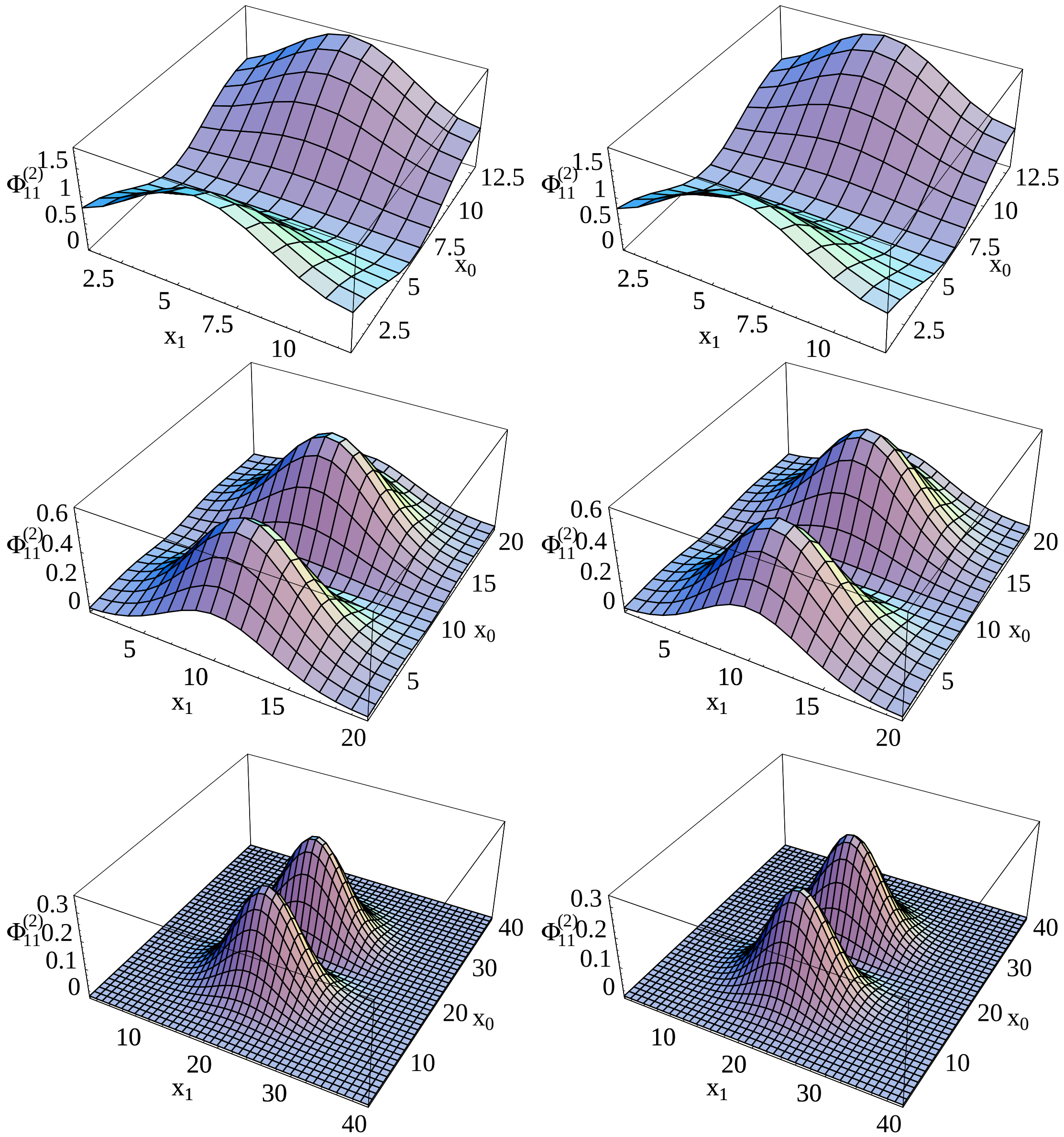

Figure 2: Invariant densities $\Phi_{11}^{(2)}$ for the configurations A, B, C of table 1, from top to bottom, are shown. Plots in the left column display the analytical perturbative result, and plots in the right column display the exact (numerical) result.

Let us briefly comment some salient features of the solution. As for the density of the component $\mathbf{E}_3$ of the electric field, a hole appears overlaying the flat background supplied by the zero-order constant abelian field. The width of this structure is, as we mentioned above, proportional to $\sqrt{l_s l_t}$, as can be derived from an analysis of the perturbative expression for the potential, and its contribution to the total action is of order Δ (cf. eq. (3.2)). Meanwhile, the action density associated to the other components of the electric field, whose contribution to the total action is as well of order Δ in the perturbative approach, exhibits in the asymmetric torus case

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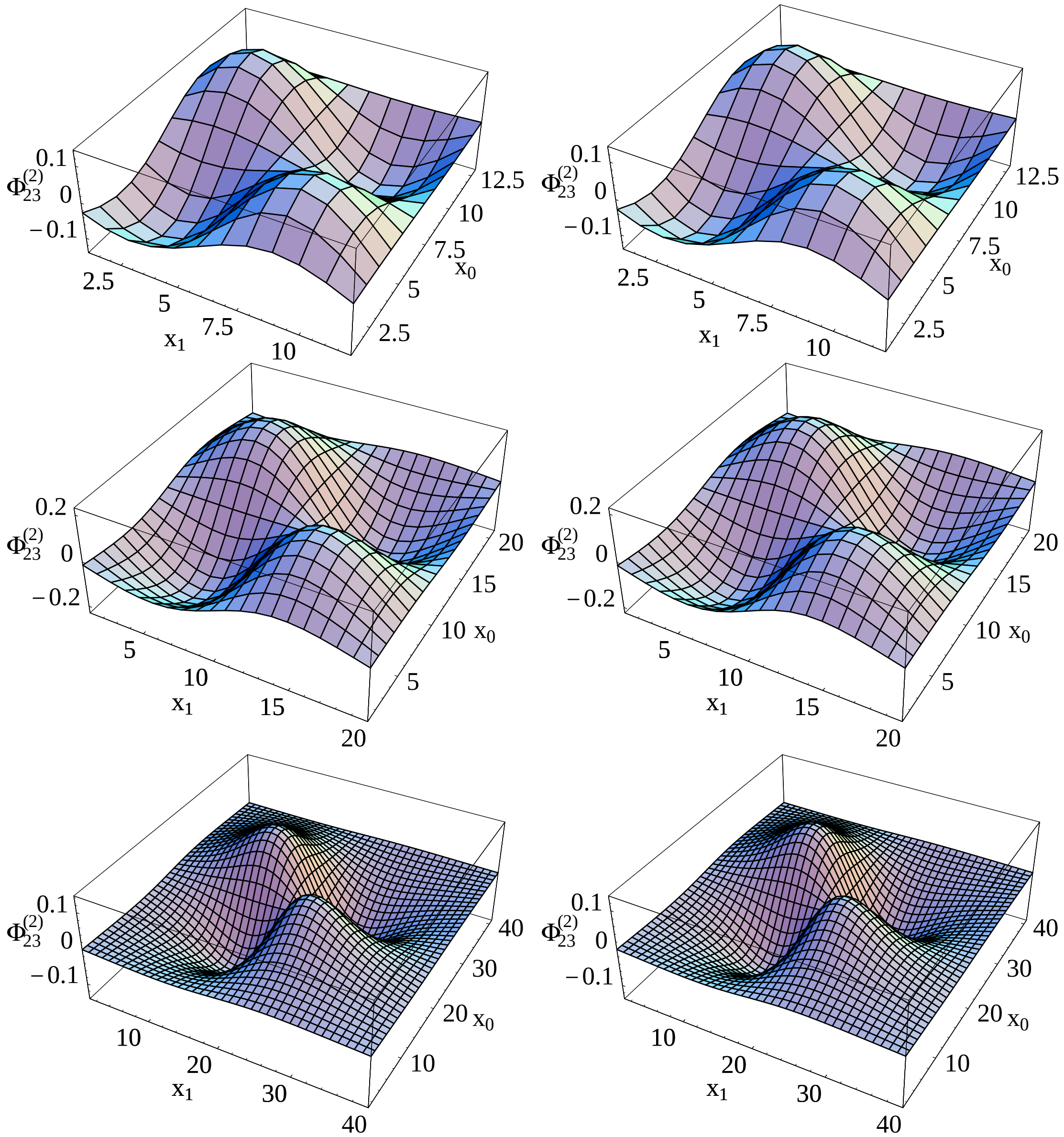

Figure 3: Invariant densities $\Phi^{(2)}_{23}$ for the configurations A, B, C of table 1, from top to bottom, are shown. Plots in the left column display the analytical perturbative result, and plots in the right column display the exact (numerical) result.

(configurations B and C) a double lump structure, again of size $\sqrt{l_s l_t}$, with the maxima aligned in euclidean time. We see that the perturbative approximation is most accurate for the configuration A, despite the relatively large value of Δ associated to it. This fact indicates that the convergence behaviour of the perturbative series depends on the value of the asymmetry parameter l_s/l_t, in such a way that it is worse the more asymmetric the torus is chosen. The overall conclusion is, anyway, that for values of Δ in the probed range, between 0.02 and 0.09, the NLO perturbative result constitutes a good approximation to the exact solution.

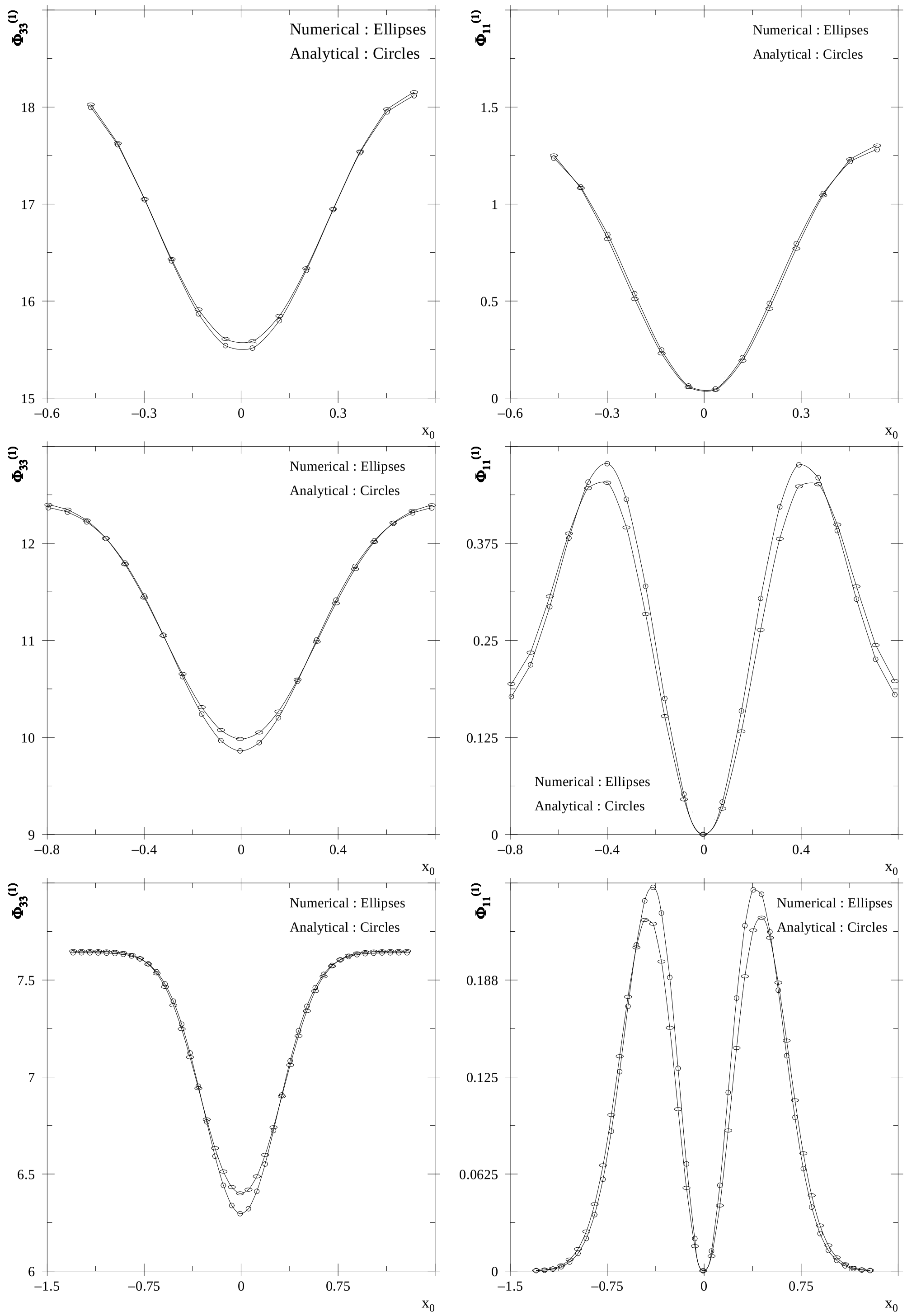

Figure 4: Comparison of exact and perturbative results for the electric field euclidean time profiles $\Phi_{33}^{(1)}$ (left column) and $\Phi_{11}^{(1)}$ (right column) in the configurations A, B, C, from top to bottom, of table 1. The x_0 coordinate has been rescaled with the lattice spacing a as defined in the text, and the center of the object has been set as origin of coordinates. The interpolating lines are intended to guide the eye.

Once the convergence behaviour of the perturbative series for small values of Δ has been checked to be good, it would be also interesting to study to what extent the NLO result at hand remains useful to describe solutions occurring at larger values of Δ. This possibility is tempting because it would open the door to apply our results to improve the analytical control over some particularly interesting fields. For instance, it is known [9, 10] that in a torus of geometry $l_t \times l_s^3$ with $l_s/l_t \ll 1$ the solutions approach self-dual fields on $T^3 \times \mathbb{R}$, the approximation being already remarkably good for $l_t \sim 3l_s$. For the considered twist and SU(2) gauge group a $Q = 1/2$ solution is obtained, whose action density displays a single lump exponentially decaying in the large direction of length l_t, and whose width is controlled by l_s. In this geometry, and setting $l_t \equiv l_s(1+\delta)$, one has $\Delta = \delta/\sqrt{1+\delta}$, and the analysis would proceed by moving from the case $\delta = 0$, where the torus is symmetric and the solution is the abelian one, to values of $\delta \sim 1$, where the features of the $T^3 \times \mathbb{R}$ solution would start to arise. Having performed computations on lattices of sizes $L_t \times L_s^3$, with $L_s = 12$ and L_t ranging from 13 to 48, we have found that the self-dual configuration evolves smoothly with changing δ. Unfortunately, the NLO perturbative approximation begins to deviate substantially from the exact result before the interesting regime $\delta \gtrsim 1$ is reached.

In the same spirit one could investigate other torus geometries, e.g. $l_t^2 \times l_s^2$ with $l_s/l_t \ll 1$, which in some cases is known to lead to limiting $T^2 \times \mathbb{R}^2$ solutions with a vortex-like structure [3], or $l_t \times l_s^3$ with $l_s/l_t \gg 1$, which leads to the $\mathbb{R}^3 \times S^1$ caloron solutions [19]. In these cases, we would expect a similar behaviour to that found for the $T^3 \times \mathbb{R}$ case.

4. Nahm transform

Nahm's transformation [15, 16] maps self-dual configurations on the torus onto other self-dual configurations. The modifications necessary to cope with twisted boundary conditions have only been worked out recently [20, 21]. In general, the transformation changes the twist tensor and torus sizes and maps the rank of the group (N) and the topological charge (Q) onto each other through the formula:

$$Q \longrightarrow Q' = N/N_0 \tag{4.1}$$

$$N \longrightarrow N' = QN_0\,, \tag{4.2}$$

which preserves the dimensionality of the moduli spaces ($QN = Q'N'$). The integer constant N_0 depends on the twist. This transformation provides an interesting tool for studying (anti-)self-dual gauge fields on the torus.

In previous sections we have expressed certain self-dual potentials as an expansion in the parameter Δ. It is henceforth interesting to analyse the interplay of this result with the Nahm transform. First of all we should find out the general properties of the Nahm transform for the configurations in question. In our case the group is SU(2) and the configuration has nontrivial twist tensor $n_{03} = n_{12} = 1$. This implies that the parameter $N_0 = 4$. Furthermore, the topological charge Q of these configurations is determined by the twist matrices and equals $Q = 1/2$, as for the corresponding constant field strength configuration. Hence, according to the formulas given above, the Nahm dual is again an SU(2) solution with topological charge $Q' = 1/2$. Now we can make use of the results of ref. [20] to determine the twist tensor and torus size of the Nahm transformed field. Indeed, the Nahm dual twist tensor is equal to the original one, and the torus size is given by: $1/2l_0 \times 1/2l_1 \times 1/2l_2 \times 1/2l_3$. Thus, except for the different torus size, the Nahm transformed field is of the same type as the original one. Furthermore, the size parameter Δ' of the Nahm transformed field is given by:

$$\Delta' = -\Delta\,. \tag{4.3}$$

Therefore, the Nahm transform provides a nonlinear relation for our perturbative expansion. A full analysis of this point is difficult and lengthy and will be left out from this paper, however it is instructive to look at the first few terms of this connection.

In order to construct the Nahm transform one has to study the zero modes of the Weyl equation in the fundamental representation of the group:

$$(\overline{\mathcal{D}} - 2\pi\imath\bar{z})\,\chi(x;z) = 0\,, \tag{4.4}$$

where z_μ represent the coordinates of a point in the Nahm dual torus. The Weyl operator $\overline{\mathcal{D}}$ contains the self-dual potential eq. (2.7) which can be expanded in powers of $\sqrt{\lambda\Delta}$. Although for $\lambda \neq 1$ the configuration is not self-dual, it is still possible to define a Nahm transform (which will not be self-dual). Thus, we can expand $\chi(x;z)$ in the same way and equate to zero all of the powers of the equation separately. From eqs. (2.21), (2.22) it is easy to see that the upper and lower components in colour space only mix for odd-even or even-odd powers of the expansion parameter. Thus, we might write:

$$\chi(x;z) = \begin{pmatrix} \phi_+ + \sqrt{\lambda\Delta}\phi'_+ \\ \phi_- + \sqrt{\lambda\Delta}\phi'_- \end{pmatrix}. \tag{4.5}$$

In the previous formula, the explicit vector is in colour space, while the quantities $\phi_\pm(x;z)$, $\phi'_\pm(x;z)$ are bi-spinors, which can be expanded in power series in $\lambda\Delta$. Eq. (4.4) amounts for ϕ_+ and ϕ'_- to the equations:

$$\begin{aligned} \left(\overline{D}_{1/2} - \imath S^\dagger - 2\pi\imath\bar{z}\right)\phi_+ &= \imath\sqrt{\lambda\Delta}W_c^\dagger\phi'_- \\ \left(\overline{D}_{-1/2} + \imath S^\dagger - 2\pi\imath\bar{z}\right)\phi'_- &= \imath\frac{1}{\sqrt{\lambda\Delta}}W^\dagger\phi_+ \end{aligned} \tag{4.6}$$

and a similar equation holds for the remaining components. The symbol $\overline{D}_{1/2}$ is defined in (2.41). We see that in this way we get two independent solutions of (4.4) as predicted by the index theorem. The Nahm transformed SU(2) vector potential is then given by the formula:

$$\widehat{A}^{ij}_\mu(z) = \imath \int d^4x\, \chi^{i\dagger}(x;z)\frac{\partial}{\partial z_\mu}\chi^j(x;z)\,, \tag{4.7}$$

where the indices i,$j \in \{1,2\}$ label the two linearly independent and orthonormal solutions.

For the whole construction, the question of the boundary conditions satisfied by the spinors is crucial. Indeed, the naive periodicity requirement:

$$\phi_\pm(x+e_\mu;z) = \exp\left\{\pm\imath\frac{\pi}{2}n_{\mu\nu}\frac{x_\nu}{l_\nu}\right\}\phi_\pm(x;z) \tag{4.8}$$

is inconsistent. How to remedy this situation is what is studied in ref. [20]. In the case at hand the easiest way out is to impose the periodicity requirement only for the x_0 and x_1 direction, while requiring only double period conditions on the other two. In short, this is just replicating the torus in the x_3 and x_2 directions. Consistently the integration in (4.7) has to be performed in this larger torus.

To illustrate the procedure we will explicitly work out the lowest order term $\phi_+^{(0)}$, which satisfies:

$$\left(\overline{D}_{1/2} - 2\pi\imath\bar{z}\right)\,\phi_+^{(0)} = 0 \tag{4.9}$$

This is a modification of the general equation studied in the previous chapter for $q=1/2$. Hence, following the same steps as before and imposing the new boundary conditions, we arrive at a unique solution (up to a multiplicative constant):

$$\phi_+^{(0)}(x;z) = \exp\{\pi\imath z_\mu y_\mu\}\widehat{\Psi}(y)\begin{pmatrix} K'^{(0)} \\ 0 \end{pmatrix}, \tag{4.10}$$

where we have defined the auxiliary variable:

$$y_\mu = x_\mu + 2l_\mu l_\nu n_{\mu\nu} z_\nu\,, \tag{4.11}$$

and the function $\widehat{\Psi}$ is the same one that appears in expression (2.51), with the replacement of $l_{2,3}$ by $2l_{2,3}$ and $\tau_{0,1}$ by $2\tau_{0,1}$. The constant $K'^{(0)}$ is fixed by the normalisation condition. In the same way one derives for $\phi_-^{(0)}$ the expression:

$$\phi_-^{(0)}(x;z) = i\tau_2\phi_+^{(0)\,*}(x;-z)\,. \tag{4.12}$$

Now we might compute $\widehat{A}^{11}_\mu(x)$ by replacing eq. (4.10) in (4.7). Now introducing the complex variables:

$$v_\mu = \frac{1}{l_\mu}(y_\mu + \imath n_{\mu\nu}y_\nu) \tag{4.13}$$

we can express all derivatives with respect to z_μ in terms of derivatives with respect to v_0, v_1, v_0^* and v_1^*. For example:

$$\frac{\partial}{\partial z_0}\phi_+^{(0)} = \left(\pi\imath y_0 - 2\imath l_3\left(\frac{\partial}{\partial v_0} - \frac{\partial}{\partial v_0^*}\right)\right)\phi_+^{(0)}\,. \tag{4.14}$$

$\widehat{\Psi}(y)$ has a very simple dependence on v_0^* and hence one has:

$$\frac{\partial}{\partial v_0^*}\phi_+^{(0)} = l_0\left(-\frac{\pi}{4l_3}v_0 + \frac{\imath\pi}{2}\left(z_0 + \imath z_3\right)\right)\phi_+^{(0)}\,. \tag{4.15}$$

The result of differentiating with respect to v_0 is much more complicated, involving derivatives of Riemann's theta function. However, in the expression for $\widehat{A}$ it is possible to integrate by parts and make the derivatives with respect to v_0 act onto the complex conjugate of $\phi_+^{(0)}$, for which the complex conjugate of eq. (4.15) allows us to obtain a simple expression. We end up with:

$$\begin{aligned}(\widehat{A}_0^{(0)})^{11}(z) &= \imath\int d^4x\ \phi_+^{(0)\dagger}\left(\pi\imath y_0 + 4\imath l_3 l_0\,\mathrm{Re}\left\{-\frac{\pi}{4l_3}v_0 + \frac{\imath\pi}{2}(z_0+\imath z_3)\right\}\right)\phi_+^{(0)}\\ &= 2\pi l_0 l_3 z_3\,.\end{aligned} \tag{4.16}$$

In the right hand side of the first equality we have displayed the contribution of the two terms entering the right hand side of (4.14). One can compute the other components of the Nahm transformed field in the same fashion arriving at:

$$\widehat{A}_\mu^{(0)}(z) = -B'_\mu(z) \tag{4.17}$$

where $B'_\mu(z)$ is given by the same expression (2.5) as in section 2, but with the lengths of the torus l_μ replaced by those of the Nahm dual one $l'_\mu = 1/2l_\mu$. The previous result implies that the Nahm dual of a constant field strength configuration is a constant field strength configuration, even if they are not self-dual, generalising the result of ref. [22].

5. Conclusions

In the previous sections we have presented a systematic expansion which allows the construction of self-dual SU(2) Yang-Mills solutions with twist tensor $n_{03} = n_{12} = 1$ on the torus. The size of higher order corrections depends on the lengths of the torus. For certain torus sizes the solution becomes equal to the well-known constant field-strength ones. The magnitude of higher order corrections grows as we move away from these torus sizes. We have also compared the landscape of the solution, as obtained from our analytical expressions to leading non-trivial order in the expansion, to the numerical result obtained through standard techniques. The result is quite satisfactory at a qualitative and quantitative level. Finally, the expansion is used to

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obtain the Nahm transform of the self-dual configuration. Curiously, the perturbative construction of the Nahm transform has the same structure as the direct perturbative construction of the self-dual configuration on the Nahm-self-dual torus. We have not been able to equate these expansions order by order, but have shown this to be the case to lowest order.

Let us now comment about the usefulness of our programme. Up to a proof, which we do not give, of convergence of our expansion, our method gives a direct proof of the existence of the solutions and for them to have the correct number of degrees of freedom. Even if calculating higher orders of the expansion turns out to be a difficult task, the expansion can be of theoretical interest for different reasons. For example, it might allow to investigate some general properties of the solutions. The case of the interplay with the Nahm transform is interesting, and should be pursued. Furthermore, one can study certain extreme limits of the torus sizes, which might allow to obtain exact solutions.

Finally, we comment that our choice of SU(2) and of the aforementioned twist tensor has been dictated by simplicity. There is, however, no a priori essential difficulty in generalising the construction given here to other SU(N) groups and different twist tensors.

Acknowledgments

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References

[1] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett.* **B 59** (1975) 85.

[2] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[3] A. González-Arroyo and A. Montero, *Self-dual vortex-like configurations in* SU(2) *Yang-Mills theory*, *Phys. Lett.* **B 442** (1998) 273 [hep-th/9809037].

[4] A. González-Arroyo, *Yang-Mills fields on the 4-dimensional torus, 1. Classical theory*, in *Advanced school for nonperturbative quantum field physics*, M. Asorey and A. Dobado eds., World Scientific 1998 [hep-th/9807108].

[5] G. 't Hooft, *A property of electric and magnetic flux in nonabelian gauge theories*, *Nucl. Phys.* **B 153** (1979) 141; *Confinement and topology in non-abelian gauge theories Acta Physica Aust. Suppl.* XXII (1980) 531.

[6] A. González-Arroyo and P. Martínez, *Investigating Yang-Mills theory and confinement as a function of the spatial volume*, *Nucl. Phys.* **B 459** (1996) 337 [hep-lat/9507001];
A. González-Arroyo, P. Martínez and A. Montero, *Gauge invariant structures and confinement*, *Phys. Lett.* **B 359** (1995) 159 [hep-lat/9507006];
A. González-Arroyo and A. Montero, *Do classical configurations produce confinement?*, *Phys. Lett.* **B 387** (1996) 823 [hep-th/9604017].

[7] G. 't Hooft, *Some twisted selfdual solutions for the Yang-Mills equations on a hypertorus*, *Comm. Math. Phys.* **81** (1981) 267.

[8] P. van Baal, SU(N) *Yang-Mills solutions with constant field strength on* T^4, *Comm. Math. Phys.* **94** (1984) 397.

[9] M. García Pérez, A. González-Arroyo and B. Söderberg, *Minimum action solutions for* SU(2) *gauge theory on the torus with nonorthogonal twist*, *Phys. Lett.* **B 235** (1990) 117;
M. García Pérez and A. González-Arroyo, *Numerical study of Yang-Mills classical solutions on the twisted torus*, *J. Phys.* **A26** (1993) 2667 [hep-lat/9206016];
A. Montero, *Numerical analysis of fractional charge solutions on the torus*, *J. High Energy Phys.* **05** (2000) 022 [hep-lat/0004009].

[10] M. García Pérez, A. González-Arroyo, J. Snippe and P. van Baal, *Instantons from over-improved cooling*, *Nucl. Phys.* **B 413** (1994) 535 [hep-lat/9309009].

[11] T.C. Kraan and P. van Baal, *Exact T-duality between calorons and Taub-Nut spaces*, *Phys. Lett.* **B 428** (1998) 268 [hep-th/9802049]; *Periodic instantons with non-trivial holonomy*, *Nucl. Phys.* **B 533** (1998) 627 [hep-th/9805168]; *Monopole constituents inside* SU*(n) calorons*, *Phys. Lett.* **B 435** (1998) 389 [hep-th/9806034];
K.Y. Lee, *Instantons and magnetic monopoles on* $R^3 \times S^1$ *with arbitrary simple gauge groups*, *Phys. Lett.* **B 426** (1998) 323 [hep-th/9802012];
K.Y. Lee and C. Hai Lu, SU(2) *calorons and magnetic monopoles*, *Phys. Rev.* **D 58** (1998) 025011 [hep-th/9802108].

[12] M. Jardim, *Construction of doubly periodic instantons*, math.DG/9909069; *Nahm transform for doubly periodic instantons*, math.DG/9910120; *Nahm transform of doubly periodic instantons*, PhD thesis, University of Oxford, 1999.

[13] A. Montero, *Vortex configurations in the large-N limit*, *Phys. Lett.* **B 483** (2000) 309 [hep-lat/0004002].

[14] C. Ford, J.M. Pawlowski, T. Tok and A. Wipf, *ADHM construction of instantons on the torus*, hep-th/0005221.

[15] W. Nahm, *A simple formalism for the BPS monopole*, *Phys. Lett.* **B 90** (1980) 413; *Self-dual monopoles and calorons*, in *Lecture Notes in Physics* vol. 201 G. Denardo et al. eds., p. 189, Springer-Verlag, Berlin, Heidelberg, New York, 1984

[16] P.J. Braam and P. van Baal, *Nahm's transformation for instantons*, *Comm. Math. Phys.* **122** (1989) 267.

[17] P. van Baal, *Nahm gauge fields for the torus*, *Phys. Lett.* **B 448** (1999) 26 [hep-th/9811112].

[18] D. Mumford, *Tata lectures on θ*, Birkhäuser, Boston 1983.

[19] M. García Pérez, A. González-Arroyo, A. Montero and P. van Baal, *Calorons on the lattice: a new perspective*, *J. High Energy Phys.* **06** (1999) 001 [hep-lat/9903022].

[20] A. González-Arroyo, *On Nahm's transformation with twisted boundary conditions*, *Nucl. Phys.* **B 548** (1999) 626 [hep-th/9811041].

[21] M. García Pérez, A. González-Arroyo, C. Pena and P. van Baal, *Nahm dualities on the torus: a synthesis*, *Nucl. Phys.* **B 564** (2000) 159 [hep-th/9905138].

[22] P. van Baal, *Instanton moduli for $T^3 \times \mathbb{R}$*, *Nucl. Phys.* **49** *(Proc. Suppl.)* (1996) 238 [hep-th/9512223].